

# Masses and Majorana fermions in graphene

Claudio Chamon<sup>1</sup>, Chang-Yu Hou<sup>2</sup>, Christopher Mudry<sup>3</sup>,  
Shinsei Ryu<sup>4</sup>, Luiz Santos<sup>5</sup>

<sup>1</sup> Physics Department, Boston University, 590 Commonwealth Avenue, Boston, MA 02215, USA

<sup>2</sup> Instituut-Lorentz, Universiteit Leiden, P.O. Box 9506, 2300 RA Leiden, The Netherlands

<sup>3</sup> Condensed Matter Theory Group, Paul Scherrer Institute, CH-5232 Villigen PSI, Switzerland

<sup>4</sup> Department of Physics, University of California, Berkeley, California 94720, USA

<sup>5</sup> Department of Physics, Harvard University, 17 Oxford Street, Cambridge, Massachusetts 02138, USA

chamon@bu.edu, hou@lorentz.leidenuniv.nl, christopher.mudry@psi.ch, sryu@berkeley.edu, santos@physics.harvard.edu

## Abstract.

We review the classification of all the 36 possible gap-opening instabilities in graphene, i.e., the 36 relativistic masses of the two-dimensional Dirac Hamiltonian when the spin, valley, and superconducting channels are included. We then show that in graphene it is possible to realize an odd number of Majorana fermions attached to vortices in superconducting order parameters if a proper hierarchy of mass scales is in place.

## 1. Introduction

The discovery [1] that it is possible to peel and isolate individual atomic layers of graphite, i.e., graphene, has led to an explosion of experimental and theoretical works and ideas from exotic physics to real material applications [2]. Graphene is a material with remarkable physical properties, many of which are consequence of its band structure at charge neutrality, which is characterized by two Fermi points in the Brillouin zone. An excitation around those Fermi points, being linearly proportional to its momentum, resembles the dispersion relation of massless relativistic particles. The low energy theory is then well described by a relativistic Dirac Hamiltonian in two dimensions [3]. Massless Dirac fermions describe graphene, but one may wonder what kinds of mass gaps could be induced. This question is rather important for device applications, as the presence of a gap would make graphene behave as a semiconductor, just like silicon. But the question is also interesting for fundamental physics reasons, as we discuss below.

A mass in the Dirac equation can be viewed as an order parameter, a bosonic field. If the mass is generated by spontaneous breaking of a symmetry – the Higgs mechanism in graphene – there can be spatio-temporal fluctuations in the mass order parameter. In particular, there could be topological defects in the order parameter: domain walls, vortices, or hedgehogs, for instance. Alternatively, the masses could be induced externally, for example if the gaps are attained by placing graphene on a certain substrate, such as one where there is a difference in the potential seen by the two atoms in the unit cell of graphene, or if the substrate is a superconductor. Topological defects could also be present in this case where the gaps are externally induced.

Topological defects in an order parameter can lead to zero modes in the fermionic spectrum, i.e., zero energy solutions lying in the middle of the gap [4]. The physical consequences of these zero modes are rather remarkable. When charge is conserved, the zero modes bind a fraction of an electron charge [5], while in superconducting systems, they bind charge neutral Majorana fermions [6].

In this paper, we classify all possible 36 competing orders of a Dirac Hamiltonian represented by 16-dimensional Dirac matrices that encode the spin, valley, and superconducting channels. We also discuss the simpler cases where only spin and valley degrees of freedom (no superconductivity), or valley alone (spinless electrons) are considered. These simpler cases serve as a warm up exercise in gaining familiarity with the classification construction as the build up of the increasingly larger representations is carried out.

We then show how this classification can be applied to the problem of selecting an odd number of Majorana fermions to bind to superconducting vortices. That the number of zero modes must be odd is important for applications to topological quantum computing, that we shall discuss in more detail below. It is possible to achieve an odd number of Majorana fermions using surface states of topological insulators [7]. But, naively, Majorana zero modes bound to superconducting vortices in graphene, first found by Ghaemi and Wilczek [6] in this context, would come in quadruplets because of the valley and spin degrees of freedom. We show otherwise. If there is a proper hierarchy of mass scales, one can tune selectively the number of Majorana zero modes in graphene from  $4 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 0$ .

We aim in this paper at a pedagogical description of the mass classification scheme, and a brief discussion of how the number of zero modes can be tuned by

changing the relative strength of masses. The original classification of masses in graphene was presented in [8], while the tuning of the number of Majorana modes was carried out in [9]. We point the reader to those papers for details beyond this review.

## 2. Classification of masses in graphene

Let us first of all define what we mean by a mass in graphene. Say we take generically a two-dimensional Dirac Hamiltonian

$$H = p_x \alpha_1 + p_y \alpha_2 + m M \quad (1)$$

where  $p_{x,y}$  stand for momentum operators in the  $x, y$ -directions, and  $\alpha_{1,2}$  are some (generically)  $D$ -dimensional matrices satisfying  $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$ . We would like to say that  $m$  is a mass scale, and that  $M$  is a mass matrix. For that to be the case, the matrix  $M$  has to satisfy certain commutation relations with the matrices  $\alpha_{1,2}$ . The relations are seen if we square the Dirac-type Hamiltonian:

$$H^2 = p_x^2 + p_y^2 + m^2 M^2 + m p_x \{\alpha_1, M\} + m p_y \{\alpha_2, M\}, \quad (2)$$

which if

$$M^2 = 1 \quad \text{and} \quad \{\alpha_{1,2}, M\} = 0, \quad (3)$$

yields

$$H^2 = p_x^2 + p_y^2 + m^2. \quad (4)$$

Thus a mass matrix is one that satisfies the relations in Eq. (3) and leads to the dispersion for a massive Dirac particle  $\varepsilon_{\mathbf{p}} = \pm \sqrt{|\mathbf{p}|^2 + m^2}$ .

Let us then count the number of such matrices for cases of increasing complexity, starting with the case of spinless fermions, marching along to the most general, when we take into account both spin and particle-hole gradings needed to discuss superconducting graphene.

### 2.1. Spinless case – 4-dimensional Dirac matrices

Let us write the  $4 \times 4$  Dirac matrices that describe spinless electrons in graphene in the Weyl representation:

$$\alpha_i \equiv \begin{pmatrix} \tau_i & 0 \\ 0 & -\tau_i \end{pmatrix} \equiv \sigma_3 \otimes \tau_i, \quad \beta \equiv \begin{pmatrix} 0 & \tau_0 \\ \tau_0 & 0 \end{pmatrix} \equiv \sigma_1 \otimes \tau_0, \quad (5)$$

where the  $2 \times 2$  unit matrix  $\tau_0$  and the three Pauli matrices  $\tau_1, \tau_2$ , and  $\tau_3$  act on the sublattices indices (A,B), while the  $2 \times 2$  unit matrix  $\sigma_0$  and the three Pauli matrices  $\sigma_1, \sigma_2$ , and  $\sigma_3$  act on the valley indices  $(+, -)$ . In this representation, the four-component spinor is given by  $\Psi^\dagger \equiv (\psi_{+B}^\dagger, \psi_{+A}^\dagger, \psi_{-A}^\dagger, \psi_{-B}^\dagger)$ .

We take the matrices  $\alpha_{1,2}$  to construct the kinetic part of the Hamiltonian, such that the gapless case can be written as

$$\mathcal{K}_0 = p_x \alpha_1 + p_y \alpha_2. \quad (6)$$

Now, the possible masses in the 4-dimensional representation of the Dirac Hamiltonian corresponds to all matrices  $M$  of the form  $X_{\mu\nu} \equiv \sigma_\mu \otimes \tau_\nu$  (other than  $\alpha_1 \equiv X_{31}$  and  $\alpha_2 \equiv X_{32}$ ) such that  $\{X_{\mu\nu}, \alpha_1\} = 0$  and  $\{X_{\mu\nu}, \alpha_2\} = 0$ . The list of such matrices has

4 elements:  $X_{33}$ ,  $X_{10}$ ,  $X_{20}$ , and  $X_{03}$ . These are then the 4 possible mass terms in spinless graphene.

Physically, we can identify these four mass matrices as follows. One perturbation that opens a gap  $2\mu_s$  is a staggered chemical potential, taking values  $+\mu_s$  and  $-\mu_s$  in the two sublattices A and B of graphene. This is the mass term added by Semenoff [10], and it corresponds to  $X_{33}$ . A second mass gap  $2|\eta|$  arises by adding directed next-nearest-neighbor hopping amplitudes in the presence of fluxes, but such that no net magnetic flux threads a hexagonal Wigner-Seitz unit cell of graphene. This perturbation, introduced by Haldane [11], breaks time-reversal symmetry (TRS) and corresponds to  $X_{03}$ . Finally, a real-valued modulation of the nearest-neighbor hopping amplitude with a wave vector connecting the two Dirac points (i.e., a Kekulé dimerization pattern for graphene) also opens a gap  $2|\Delta|$  [5]. This real-valued modulation of the nearest-neighbor hoppings is parametrized by the complex order parameter  $\Delta = \text{Re } \Delta + i \text{Im } \Delta$  whose phase controls the angles of the dimerization pattern. This complex order parameter translates into two real masses  $\text{Re } \Delta$  and  $\text{Im } \Delta$ , corresponding respectively to  $X_{10}$  and  $X_{20}$ , bringing the total number of real-valued masses to four.

We can identify the microscopic origin of these 4 masses according to the symmetries that they respect (or break) at the level of the Dirac equation. To this end, let us look into two symmetries that the Dirac Hamiltonian  $H$  may or may not possess:

- Time-reversal symmetry (TRS) is satisfied when

$$X_{11} H^*(-p) X_{11} = H(p) . \quad (7)$$

- Sub-lattice symmetry (SLS), also referred to as chiral symmetry in the literature, is satisfied when

$$X_{33} H(p) X_{33} = -H(p) . \quad (8)$$

This symmetry operation corresponds to flipping the sign of the wavefunction in one of the sublattices but not on the other (here, we flip the sign on sublattice A).

The Kekulé dimerization pattern corresponds to a spatial modulation of the hopping matrix elements between sublattices A and B, and therefore changing the sign of the wavefunction on one sublattice would reverse the overall sign of the Hamiltonian. And indeed, one verifies that masses of the form  $X_{10}$  and  $X_{20}$  satisfy Eq. (8). The hopping amplitudes in the Kekulé dimerization pattern are all real, and therefore they respect TRS, which can also be checked via Eq. (7).

Both the staggered chemical potential or Semenoff mass  $X_{33}$  and the Haldane mass  $X_{03}$  break SLS, as they involve couplings between sites in the same sublattices. However, the staggered chemical potential respects TRS, while the Haldane mass breaks it, as can be checked using Eqs. (7) and (8).

We shall introduce here a terminology that will be useful later on when we consider systems with larger size representations, once spin degrees of freedom and superconductivity are considered. For instance, the Kekulé dimerization pattern corresponds to a valence-bond solid (VBS) order parameter by analogy to the terminology used for quantum dimer models. A VBS order picks up a microscopic orientation that translates into a complex-valued order parameter in the continuum limit. Hence, we shall distinguish between the real (ReVBS) and imaginary (ImVBS)

Mass matrix	Order parameter	TRS	SLS
$X_{10}$	ReVBS	True	True
$X_{20}$	ImVBS	True	True
$X_{33}$	CDW	True	False
$X_{03}$	QHE	False	False

**Table 1.** The 4 mass matrices that can be added to the massless Dirac Hamiltonian  $\mathcal{K}_0$  from Eq. (6) are of the form  $X_{\mu\nu} \equiv \sigma_\mu \otimes \tau_\nu$  and anticommute with  $\mathcal{K}_0$ . Each mass matrix can be assigned an order parameter for the underlying microscopic model. Each mass matrix preserves or breaks time-reversal symmetry (TRS), see Eq. (7) and sublattice symmetry (SLS), see Eq. (8).

parts of the VBS, and here they correspond to the  $\text{Re}\Delta$  and  $\text{Im}\Delta$  of the Kekulé distortion. Now, any mass matrix that does not satisfy the SLS defined in Eq. (8) corresponds to a microscopic order parameter for which the fermion bilinear has the two lattice fermions sitting on the same sublattice. This is the case of the mass associated to the staggered chemical potential, which we may identify with a charge-density wave (CDW). Finally, the Haldane mass implies a quantum Hall effect (QHE). This nomenclature, along with the classification of the 4 masses according to TRS and SLS, are summarized in Table 1.

## 2.2. Spinful case – 8-dimensional Dirac matrices

Having warmed up with the  $4 \times 4$  representations of the Dirac matrices for the simpler spinless case, we now construct the representations of all the masses for the case of graphene with the spin degrees of freedom included, but still no superconductivity considered.

To represent the single particle Hamiltonian  $\mathcal{K}$ , utilize the 64 8-dimensional Hermitean matrices

$$X_{\mu_1\mu_2\mu_3} \equiv s_{\mu_1} \otimes \sigma_{\mu_2} \otimes \tau_{\mu_3} \quad (9)$$

where  $\mu_{1,2,3} = 0, 1, 2, 3$ . Here, we have introduced the three families  $s_{\mu_1}$ ,  $\sigma_{\mu_2}$ , and  $\tau_{\mu_3}$  of unit  $2 \times 2$  and Pauli matrices that encode the spin-1/2, valley, and sublattice degrees of freedom of graphene, respectively.

The masses should be added to the massless Dirac Hamiltonian

$$\mathcal{K}_0 = p_x \alpha_1 + p_y \alpha_2, \quad \text{where } \alpha_1 \equiv X_{031} \text{ and } \alpha_2 \equiv X_{032}. \quad (10)$$

The possible masses in the 8-dimensional representation of the Dirac Hamiltonian correspond to all matrices  $M$  of the form  $X_{\mu_1\mu_2\mu_3}$  (other than  $\alpha_1$  and  $\alpha_2$ ) such that  $\{X_{\mu_1\mu_2\mu_3}, \alpha_1\} = 0$  and  $\{X_{\mu_1\mu_2\mu_3}, \alpha_2\} = 0$ . One can carry out the exercise of finding such matrices, obtaining the 16 matrices listed in Table 2.

We can classify these matrices according to the symmetries that they respect (or break). The symmetries that the Hamiltonian  $H$  may possess are as follows:

- Time-reversal symmetry (TRS) is satisfied when

$$X_{211} H^*(-p) X_{211} = H(p). \quad (11)$$

- Sub-lattice symmetry (SLS) is satisfied when

$$X_{033} H(p) X_{033} = -H(p) . \quad (12)$$

- Spin-rotation symmetry (SRS) is satisfied when

$$[X_{100}, H(p)] = [X_{200}, H(p)] = [X_{300}, H(p)] = 0 . \quad (13)$$

Notice that once we introduce spin degrees of freedom, we can decide whether the Hamiltonian is spin rotational invariant or not. Also, we can introduce terminology similar to the VBS and CDW cases we used in labeling the masses for the spinless case. Masses that do not satisfy the SLS defined in Eq. (12) correspond to a microscopic order parameters for which the fermion bilinear has the two lattice fermions sitting on the same sublattice. Microscopic examples are, in addition to the CDW already previously encountered, the spin-density waves (SDW) such as Néel ordering, orbital currents leading to the quantum Hall effect (QHE), and spin-orbit couplings leading to the quantum spin Hall effect (QSHE). Whenever the instability can have a direction associated to it in internal spin space, we add the corresponding directional subscript  $x, y, z$ .

The first 4 masses listed in Table 2 are physically the same as those 4 listed in Table 1 for the case of spinless electrons. These 4 masses correspond to order parameters in the charge sector. The next 12 masses correspond to some form of magnetic order. The simpler are the Néel $_{x,y,z}$  order parameters along the three directions. The Néel states are the SDW order associated to fermion bilinears at the same lattice site. The ReVBS $_{x,y,z}$  and ImVBS $_{x,y,z}$  are their counterparts where the fermion bilinears are defined on the bonds instead of the sites. Finally, the QSHE $_{x,y,z}$  correspond to the quantum spin Hall effect discussed by Kane and Mele in Ref. [12].

### 2.3. Fully general case of single-layer graphene – 16-dimensional Dirac matrices

To describe all symmetry-breaking instabilities with a local order parameter in graphene we consider the Bogoliubov-de Gennes (BdG) Hamiltonian

$$\hat{H}_{\text{BdG}} = \frac{1}{2} \int d^2 \mathbf{r} \hat{\Psi}^\dagger H \hat{\Psi} \quad (14)$$

where  $\hat{\Psi}$  is the 16-component Nambu spinor

$$\hat{\Psi} \equiv \left( \hat{\psi}_\uparrow^\dagger, \hat{\psi}_\downarrow^\dagger, \hat{\psi}_\uparrow, \hat{\psi}_\downarrow \right)^t \quad (15)$$

and  $\hat{\psi}_{s=\uparrow,\downarrow}$  is a 4-component fermion annihilation operator that accounts for the 2 valley and the 2 sublattice degrees of freedom. The kernel of the BdG Hamiltonian has the block structure

$$H = \begin{pmatrix} \mathcal{H}_{\text{pp}} & \mathcal{H}_{\text{ph}} \\ \mathcal{H}_{\text{ph}}^\dagger & -\mathcal{H}_{\text{pp}}^t \end{pmatrix} \quad (16)$$

where the  $8 \times 8$  blocks  $\mathcal{H}_{\text{pp}}$  and  $\mathcal{H}_{\text{ph}}$  act on the combined space of valley, sublattice, and spin degrees of freedom, and represent the normal and anomalous part of the BdG Hamiltonian, respectively. These blocks satisfy

$$\mathcal{H}_{\text{pp}}^\dagger = \mathcal{H}_{\text{pp}} \quad (\text{Hermiticity}), \quad \mathcal{H}_{\text{ph}}^t = -\mathcal{H}_{\text{ph}} \quad (\text{Fermi statistics}). \quad (17)$$

To represent the single particle Hamiltonian  $H$ , define the 256 16-dimensional Hermitean matrices

$$X_{\mu_1 \mu_2 \mu_3 \mu_4} \equiv \rho_{\mu_1} \otimes s_{\mu_2} \otimes \sigma_{\mu_3} \otimes \tau_{\mu_4} \quad (18)$$

Mass matrix	Order parameter	TRS	SRS	SLS
$X_{010}$	ReVBS	True	True	True
$X_{020}$	ImVBS	True	True	True
$X_{033}$	CDW	True	True	False
$X_{003}$	QHE	False	True	False
$X_{110}$	ReVBS <sub><i>x</i></sub>	False	False	True
$X_{210}$	ReVBS <sub><i>y</i></sub>	False	False	True
$X_{310}$	ReVBS <sub><i>z</i></sub>	False	False	True
$X_{120}$	ImVBS <sub><i>x</i></sub>	False	False	True
$X_{220}$	ImVBS <sub><i>y</i></sub>	False	False	True
$X_{320}$	ImVBS <sub><i>z</i></sub>	False	False	True
$X_{103}$	QSHE <sub><i>x</i></sub>	True	False	False
$X_{203}$	QSHE <sub><i>y</i></sub>	True	False	False
$X_{303}$	QSHE <sub><i>z</i></sub>	True	False	False
$X_{133}$	Néel <sub><i>x</i></sub>	False	False	False
$X_{233}$	Néel <sub><i>y</i></sub>	False	False	False
$X_{333}$	Néel <sub><i>z</i></sub>	False	False	False

**Table 2.** The 16 mass matrices that can be added to the massless Dirac Hamiltonian  $\mathcal{K}_0$  from Eq. (10) are of the form  $X_{\mu\nu\lambda} \equiv s_\mu \otimes \sigma_\nu \otimes \tau_\lambda$  and anticommute with  $\mathcal{K}_0$ . Each mass matrix can be assigned an order parameter for the underlying microscopic model. The latin subindex of the order parameter's name corresponds to the preferred quantization axis in SU(2) spin space. Each mass matrix preserves or breaks time-reversal symmetry (TRS), see Eq. (11), spin-rotation symmetry (SRS), see Eq. (12), and sublattice symmetry (SLS), see Eq. (13).

where  $\mu_{1,2,3,4} = 0, 1, 2, 3$ . Here, we have introduced the four families  $\rho_{\mu_1}$ ,  $s_{\mu_2}$ ,  $\sigma_{\mu_3}$ , and  $\tau_{\mu_4}$  of unit  $2 \times 2$  and Pauli matrices that encode the particle-hole (Nambu), spin-1/2, valley, and sublattice degrees of freedom of graphene, respectively.

The Dirac kinetic energy  $\mathcal{K}_0$  of graphene that accounts for the BdG block structure (16) is given by

$$\mathcal{K}_0 = p_x \alpha_1 + p_y \alpha_2, \quad \text{where} \quad \alpha_1 \equiv X_{0031} \quad \text{and} \quad \alpha_2 \equiv X_{3032}. \quad (19)$$

There are  $64 = 4 \times 16$  mass matrices (i.e.,  $X_{\mu_1\mu_2\mu_3\mu_4}$  that anticommutes with  $\mathcal{K}_0$ ). Of these 64 mass matrices, only 36 satisfy the condition

$$X_{1000} X_{\mu_1\mu_2\mu_3\mu_4}^\dagger X_{1000} = -X_{\mu_1\mu_2\mu_3\mu_4} \quad (20)$$

for particle-hole symmetry (PHS) and are thus compatible with the symmetry condition  $(\rho_1 \otimes s_0 \otimes \sigma_0 \otimes \tau_0 \hat{\Psi})^\dagger = \hat{\Psi}^\dagger$  on the Nambu spinors [i.e., compatible with Eq. (17)]. All mass matrices with PHS are enumerated in Table 3.

All 36 mass matrices from Table 3 can be classified in terms of the following (microscopic) 3 symmetry properties.

- A BdG Hamiltonian has time-reversal symmetry (TRS) when

$$X_{0211} H^*(-p) X_{0211} = H(p). \quad (21)$$

- A BdG Hamiltonian has sublattice symmetry (SLS) when

$$X_{0033} H(p) X_{0033} = -H(p). \quad (22)$$

- A BdG Hamiltonian has SU(2) spin rotation symmetry (SRS) when

$$[X_{3100}, H(p)] = [X_{0200}, H(p)] = [X_{3300}, H(p)] = 0. \quad (23)$$

Identifying the physical meaning of the masses is done in a similar way as explained in the simpler cases discussed previously (the spinless and spinful cases without superconductivity). Below we present the rational for the terminology in full generality.

The microscopic order parameter corresponding to a mass matrix satisfying the SLS defined in Eq. (22) is a non-vanishing expectation value for a fermion bilinear with the two lattice fermions residing on the ends of a bond connecting sites in opposite sublattices. We shall say that such a mass matrix is associated to a valence-bond solid (VBS) order parameter in analogy to the terminology used for quantum dimer models. A VBS order picks up a microscopic orientation that translates into a complex-valued order parameter in the continuum limit. Hence, we shall distinguish between the real (ReVBS) and imaginary (ImVBS) parts of the VBS. Triplet superconductivity (TSC) is also possible on bonds connecting the two sublattices. The terminology TSC will then also be used. To distinguish TSC with or without TRS we shall reserve the prefixes Re and Im for real and imaginary parts. This is a different convention for the use of the prefixes Re and Im than for a VBS.

Any mass matrix that does not satisfy the SLS defined in Eq. (22) corresponds to a microscopic order parameter for which the fermion bilinear has the two lattice fermions sitting on the same sublattice. Microscopic examples are charge-density waves (CDW), spin-density waves (SDW) such as Néel ordering, orbital currents leading to the quantum Hall effect (QHE), spin-orbit couplings leading to the quantum spin Hall effect (QSHE), singlet superconductivity (SSC), or triplet superconductivity (TSC).

When SU(2) spin symmetry is broken by the order parameter, we add a subindex  $x$ ,  $y$ , or  $z$  that specifies the relevant quantization axis to the name of the mass matrix. Moreover, TSC with SLS must be distinguished by the 2 possible bond orientations (the underlying two-dimensional lattice has 2 independent vectors connecting nearest-neighbor sites). These 2 orientations are specified by the Pauli matrices used in the valley and sublattice subspaces, i.e., by the 2 pairs of numbers 02 and 32. Symmetry properties of all 36 PHS masses are summarized in Table 3.

### 3. Majorana bound states in superconducting graphene

The unconventional relativistic-like band structure of graphene leads to striking physical phenomena, for example when graphene is placed in proximity with a superconductor. The problem of two dimensional Dirac fermions coupled to an s-wave superconductor was considered by Jackiw and Rossi [4], who have shown that the fermionic spectrum displays a single zero energy mode if the superconducting order parameter winds once about a given point in space (the center of the vortex). This result is to be contrasted with the case of non-relativistic s-wave superconductors, for which no zero mode exists in the vortex core [14].



Mass matrix	Order parameter	TRS	SRS	SLS
$X_{3010}$	ReVBS	True	True	True
$X_{0020}$	ImVBS	True	True	True
$X_{3033}$	CDW	True	True	False
$X_{3003}$	QHE	False	True	False
$X_{3110}$	ReVBS <sub><i>x</i></sub>	False	False	True
$X_{0210}$	ReVBS <sub><i>y</i></sub>	False	False	True
$X_{3310}$	ReVBS <sub><i>z</i></sub>	False	False	True
$X_{0120}$	ImVBS <sub><i>x</i></sub>	False	False	True
$X_{3220}$	ImVBS <sub><i>y</i></sub>	False	False	True
$X_{0320}$	ImVBS <sub><i>z</i></sub>	False	False	True
$X_{3103}$	QSHE <sub><i>x</i></sub>	True	False	False
$X_{0203}$	QSHE <sub><i>y</i></sub>	True	False	False
$X_{3303}$	QSHE <sub><i>z</i></sub>	True	False	False
$X_{3133}$	Néel <sub><i>x</i></sub>	False	False	False
$X_{0233}$	Néel <sub><i>y</i></sub>	False	False	False
$X_{3333}$	Néel <sub><i>z</i></sub>	False	False	False
$X_{2211}$	ReSSC	True	True	False
$X_{1211}$	ImSSC	False	True	False
$X_{1002}$	ReTSC <sub>02<i>y</i></sub>	True	False	True
$X_{2002}$	ImTSC <sub>02<i>y</i></sub>	False	False	True
$X_{1102}$	ReTSC <sub>02<i>z</i></sub>	False	False	True
$X_{2102}$	ImTSC <sub>02<i>z</i></sub>	True	False	True
$X_{1302}$	ReTSC <sub>02<i>x</i></sub>	False	False	True
$X_{2302}$	ImTSC <sub>02<i>x</i></sub>	True	False	True
$X_{1032}$	ReTSC <sub>32<i>y</i></sub>	False	False	True
$X_{2032}$	ImTSC <sub>32<i>y</i></sub>	True	False	True
$X_{1132}$	ReTSC <sub>32<i>z</i></sub>	True	False	True
$X_{2132}$	ImTSC <sub>32<i>z</i></sub>	False	False	True
$X_{1332}$	ReTSC <sub>32<i>x</i></sub>	True	False	True
$X_{2332}$	ImTSC <sub>32<i>x</i></sub>	False	False	True
$X_{1021}$	ReTSC <sub><i>y</i></sub>	True	False	False
$X_{2021}$	ImTSC <sub><i>y</i></sub>	False	False	False
$X_{1121}$	ReTSC <sub><i>z</i></sub>	False	False	False
$X_{2121}$	ImTSC <sub><i>z</i></sub>	True	False	False
$X_{1321}$	ReTSC <sub><i>x</i></sub>	False	False	False
$X_{2321}$	ImTSC <sub><i>x</i></sub>	True	False	False

**Table 3.** The 36 mass matrices with particle-hole symmetry (PHS), see Eq. (20), for the massless Dirac Hamiltonian  $\mathcal{K}_0$  from Eq. (19) are of the form (18) and anticommute with  $\mathcal{K}_0$ . Each mass matrix can be assigned an order parameter for graphene. The latin subindex of the order parameter's name corresponds to the preferred quantization axis in SU(2) spin space. The pair of numeral subindices 02 and 32 are used to distinguish the two unit vectors spanning two-dimensional space. Each mass matrix preserves or breaks time-reversal symmetry (TRS), see Eq. (21), spin-rotation symmetry (SRS), see Eq. (23), and sublattice symmetry (SLS), see Eq. (22).

Because superconductivity mixes particles and holes, the second quantized operator  $\Gamma$  associated with the zero energy mode turns out to be self-adjoint, that is to say,  $\Gamma = \Gamma^\dagger$ . It is in the sense of being a “real” fermion that a zero mode represents a condensed matter realization of Majorana fermions [15, 16]. An enormous amount of interest rests upon Majorana states regarding their possible relevance to constructing topological qubits: with two spatially separated vortices, each of which hosting one zero mode, it is possible to form a two level system (qubit) that stores the information non-locally. The parity of the number of zero modes per vortex core turns out to be fundamental in determining the stability of the qubit. If, for example, two zero modes exist at each vortex, generic perturbations can split those modes apart causing the breaking down of the stability. The general statement is then that an odd (even) number of zero modes per vortex implies that one can form qubits that are stable (unstable) against decoherence.

Let us suppose now that graphene is brought near to a good conventional s-wave superconductor. By proximity effect, electron-hole pairs can tunnel between graphene and the superconductor. In this way, superconductivity can be induced in graphene. In the presence of a superconducting vortex, the low energy theory is an extension of that considered by Jackiw and Rossi. Zero mode solutions exist [6] but now there are four Majorana modes per vortex, as opposed to just an isolated Majorana mode in the Jackiw-Rossi system. The even number of zero modes is a direct consequence of the fermion doubling problem: in any TRS and local lattice theory, the number of Dirac cones is even [13]. Graphene can thus be regarded, as far as the discussion of zero modes is concerned, as four copies of the Jackiw-Rossi model.

For the purpose, at least as a question of principle, of building topological qubits in graphene, or for that matter in any lattice system with Dirac particles, it is necessary to design a mechanism by which one can control the parity of the number of zero modes bound to a singly-quantized vortex in order to overcome the serious challenge originating from the fermion doubling problem.

The same dilemma, but in a different context, is present in the implementation of lattice gauge theories, where the lattice regularization introduces spurious fermionic degrees of freedom. Wilson has proposed to overcome this problem by introducing terms “by hand” in the Hamiltonian with the effect of adding mass terms to those unwanted fermions, thus removing them from the low energy sector [17]. Although such Wilson masses indeed remove the extra fermionic particles at tree-level, these perturbations have to be treated with great caution when quantum fluctuations are taken into account.

We advocate that for some lattice systems considered in condensed matter physics, the Wilson proposal is the way to control the parity of the number of zero modes. Hereafter, we explain how the Wilson mechanism works for graphene.

We recall that in graphene, electrons with spin  $s = \uparrow, \downarrow$  hop on a honeycomb lattice that is made of two triangular sublattices A and B. The conduction and valence bands touch at the two non-equivalent points  $\mathbf{K}_\pm$  located at the opposite corners in the hexagonal first Brillouin zone (see Ref. [2] for a review). Finally, to account for the possibility of a SC instability, Nambu doublets are introduced with the index p and h to distinguish particles from their charge conjugate (holes). Hence, after linearization of the spectrum about the Fermi points  $\mathbf{K}_\pm$ , this leads to a single-particle kinetic energy represented by a  $16 \times 16$  dimensional matrix  $\mathcal{K}_0 \equiv p_x \alpha_1 + p_y \alpha_2$ . Here,  $\alpha_1$  and  $\alpha_2$  are two  $16 \times 16$  dimensional Dirac matrices.

As discussed in Sec. 2 above (and in Ref. [8]), there exists 36 distinct order

parameters (listed in Table 3) such that any one, when added to  $\mathcal{K}_0$ , opens a spectral gap. These order parameters were identified by seeking all  $16 \times 16$  matrices from the Clifford algebra that anticommute with  $\mathcal{K}_0$ . Among these order parameters, two (a real and an imaginary part) correspond to one complex-valued order parameter that is associated with singlet superconductivity. We shall denote the two corresponding  $16 \times 16$  matrices from the Clifford algebra by  $M_{\text{ReSSC}}$  and  $M_{\text{ImSSC}}$  and define the perturbation  $\mathcal{H}_\Delta \equiv \Delta_1 M_{\text{ReSSC}} + \Delta_2 M_{\text{ImSSC}}$  that opens the spectral gap  $2|\Delta|$  with the complex-valued  $\Delta \equiv \Delta_1 + i\Delta_2$  parametrized by the real-valued  $\Delta_1$  and  $\Delta_2$  when added to  $\mathcal{K}_0$ .

Next, we would like to select other order parameters among the remaining  $34(=36-2)$  masses that *compete* with superconductivity, i.e., masses that do not add in quadrature with the two superconducting masses  $M_{\text{ReSSC}}$  and  $M_{\text{ImSSC}}$ . Matrices corresponding to masses that add in quadrature anticommute, while matrices corresponding to masses that compete commute. We thus seek all  $16 \times 16$  matrices from the Clifford algebra that (i) anticommute with  $\mathcal{K}_0$  and (ii) commute with  $\mathcal{H}_\Delta$ . In this way, we find all 10 TRS-breaking order parameters listed in Table 4 that alone would open a gap in the Dirac spectrum if not for their competition with the gap induced by singlet superconductivity. Within this set of 10 matrices one can form groups of at most 4 matrices that are mutually commuting and therefore can be simultaneously diagonalized. Let us choose the 4-tuplet  $\{\text{ReVBS}_x, \text{ImVBS}_y, \text{Neel}_z, \text{IQHE}\}$  for concreteness, but the results hereafter apply to any other such 4-tuplet of commuting mass matrices among the set of 10. Observe that any member of this 4-tuplet breaks TRS. This property will allow us to overcome the fermion doubling barrier [13]. It is then possible to show that

$$\begin{aligned} \mathcal{H} = & \mathbf{p} \cdot \boldsymbol{\alpha} + \Delta_1 M_{\text{ReSSC}} + \Delta_2 M_{\text{ImSSC}} \\ & + m_1 M_{\text{ReVBS}_x} + m_2 M_{\text{ImVBS}_y} + m_3 M_{\text{Neel}_z} + \eta M_{\text{IQHE}}, \end{aligned} \quad (24)$$

after a unitary transformation, can be brought into the form

$$\mathcal{H} \equiv \begin{pmatrix} \mathcal{H}_1 & 0 & 0 & 0 \\ 0 & \mathcal{H}_2 & 0 & 0 \\ 0 & 0 & \mathcal{H}_3 & 0 \\ 0 & 0 & 0 & \mathcal{H}_4 \end{pmatrix} \quad (25)$$

with the  $4 \times 4$  Hermitean blocks

$$\mathcal{H}_j = \begin{pmatrix} -\eta_j & p & \delta_j & 0 \\ \bar{p} & \eta_j & 0 & \delta_j \\ \bar{\delta}_j & 0 & -\eta_j & -p \\ 0 & \bar{\delta}_j & -\bar{p} & \eta_j \end{pmatrix} \quad (26)$$

whereby the complex notation  $p \equiv p_1 + ip_2$  is used for the momenta and  $\bar{x}$  denotes the complex conjugate of  $x$ . It is found that

$$\delta_{1,2,3,4} \equiv \Delta, \quad (27)$$

$$\eta_1 \equiv -m_1 + m_2 + m_3 + \eta, \quad (28)$$

$$\eta_2 \equiv m_1 - m_2 + m_3 + \eta, \quad (29)$$

$$\eta_3 \equiv m_1 + m_2 - m_3 + \eta, \quad (30)$$

$$\eta_4 \equiv -m_1 - m_2 - m_3 + \eta. \quad (31)$$

Mass matrix	Order parameter	TRS	SRS	SLS
$X_{3003}$	IQHE	False	True	False
$X_{3110}$	ReVBS <sub><i>x</i></sub>	False	False	True
$X_{0210}$	ReVBS <sub><i>y</i></sub>	False	False	True
$X_{3310}$	ReVBS <sub><i>z</i></sub>	False	False	True
$X_{0120}$	ImVBS <sub><i>x</i></sub>	False	False	True
$X_{3220}$	ImVBS <sub><i>y</i></sub>	False	False	True
$X_{0320}$	ImVBS <sub><i>z</i></sub>	False	False	True
$X_{3133}$	Néel <sub><i>x</i></sub>	False	False	False
$X_{0233}$	Néel <sub><i>y</i></sub>	False	False	False
$X_{3333}$	Néel <sub><i>z</i></sub>	False	False	False

**Table 4.** The 10 mass matrices with particle-hole symmetry (PHS) that anticommute with  $\alpha_1$  and  $\alpha_2$  and commute with the singlet SC masses  $M_{\text{ReSSC}}$  and  $M_{\text{ImSSC}}$ . Each mass matrix can be assigned an order parameter for the underlying microscopic model. The latin subindex of the order parameter's name corresponds to the preferred quantization axis in SU(2) spin space. Each mass matrix either preserves or breaks time-reversal symmetry (TRS), spin-rotation symmetry (SRS), and sublattice symmetry (SLS). An explicit representation defined in Ref. [8] is given in the last column.

The breaking of the 16-dimensional matrix into four independent sectors is key to the ability of controlling the number of zero modes. The argument goes as follows. If all the  $\eta_j$ 's are zero and the SC order parameter has a single vortex with unit winding number, there are 4 Majorana fermions bound to it [6]. Indeed, in this limit one has precisely four copies of the Jackiw-Rossi model, with any one copy delivering one zero-mode.

However, as the magnitudes of the  $|\eta_j|$ 's increase, there will be a phase transition every time that  $|\eta_j| = |\Delta(r = \infty)|$ , where  $|\Delta(r = \infty)|$  is the bulk value of the order parameter far away from the center of the vortex. There is no zero-mode attached to vortices in the  $j$ -th copy when  $|\eta_j| > |\Delta(r = \infty)|$ , as can be explicitly checked. Indeed, this gapped phase is adiabatically connected to the limit  $|\Delta(r = \infty)|/|\eta_j| = 0$  with no superconductivity, i.e., no support for a zero mode (in this gapped phase, the spectral symmetry of the BdG Hamiltonian prevents any change of the parity in the number of zero modes). Therefore, one can knock out the Majorana fermions one by one by taking the values of the four  $|\eta_j|$ 's across the phase transitions.

In summary, we have identified a mechanism to overcome the fermion-doubling barrier that can prevent the attachment of an odd number of Majorana fermions to the core of SC vortices in graphene-like tight-binding models. This mechanism relies on a topological charge that measures the parity in the number of Majorana fermions attached to an isolated vortex and the use of TRS-breaking order parameters that compete with each other and with the SC order parameter to knock out one by one the Majorana fermions. Therefore one can selectively switch between even and odd numbers.

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